Sub-algebra

Let $(B, +, *, ', 0, 1)$ be a Boolean Algebra. A non-empty subset S of B is said to be a sub algebra iff S itself is a Boolean algebra w.r.t. operation $+$, $*$ and $'$ of B.

Ex:- $\{0, 1\}$ & B are sub-algebras of B.

Th:- A non-empty subset S of a Boolean algebra B is sub-algebra of B iff iff S is closed under the three operations of B i.e. $+$, $*$ and $'$.

Pf:- Let S be a sub algebra of B. Then S itself is a Boolean Algebra under $+$, $*$ & $'$ defined on B.

$\therefore$ S is closed under these three operations. Thus—

$$a, b \in S \Rightarrow a+b \in S,\ a*b \in S \ \&\ a' \in S$$

Converse Given S is closed under the operations $+$, $*$ and $'$ of B. i.e.

$$a, b \in S \Rightarrow a+b,\ a*b \ \&\ a' \in S$$

To prove that S is a sub-algebra of B.

First we shall show that $0, 1 \in S$.

Since S is non-empty

$\therefore a \in S \Rightarrow a' \in S$

Now $a \in S, a' \in S \Rightarrow a + a' \in S$ & $a * a' \in S$

$\Rightarrow 1 \in S$ & $0 \in S$

Now we shall show four axioms are satisfied for S.

1. Commutative Laws

$a, b \in S \Rightarrow a, b \in B$ as $S \subset B$

$\therefore a+b = b+a$ & $a * b = b * a$

2. Identity Laws

For any $a \in S$, we have $0, 1 \in S$ s.t.

$a+0 = a$ & $a * 1 = a$

3. Distr Laws

Since $S \subset B$ & both the binary ops '+' & '*' are distributive in B, $\therefore$ these will be distributive in S also.

4. Complement Laws

Let $a \in S$. Then $a' \in S$

s.t. $a + a' = 1$ & $a * a' = 0$

$\therefore$ S itself a Boolean Algebra & $\therefore$ a sub-algebra of B.

(Proved)

Isomorphic Boolean Algebra

Two Boolean algebras B & B' are said to be isomorphic if ∃ a bijective mapping 'f' from B onto B' s.t.

$$f(a+b) = f(a) + f(b)$$

$$f(a*b) = f(a) * f(b)$$

$$f(a') = [f(a)]' \quad \forall\, a, b \in B$$

[Note- Bijective means 1-1 & onto]

i.e. operations are preserved by 'f'.

Ex- B = {0,1}; +, * & ' are defined as-

+	0	1
0	0	1
1	1	1

*	0	1
0	0	0
1	0	1

'	0	1
	1	0

B → Boolean Algebra

B' = {a,b}; +', *', ¯ defined as -

+'	a	b
a	a	b
b	b	b

*'	a	b
a	a	a
b	a	b

¯	a	b
	b	a

B' → Boolean Algebra

Then

$$f: B \to B'$$

s.t. $f(0) = a$, $f(1) = b$

is bijective and preserves the three operations.

∴ B & B' are isomorphic to each other.

Q.4. B, B' → isomorphic

$f: B \to B'$ isomorphic mapping.

Then

(i) If 0 is the identity for '+' in B then $f(0)$ is the identity for '+' in B'

(ii) If 1 is the identity for * in B then $f(1)$ is the identity for * in B'.

Pf → (i) 0 → identity for + in B.

Let 0^* be identity for + in B'.

Then,

$f(0) = f(a * a')$, $a * a' = 0$

$= f(a) * f(a')$, f isomorphic

$= f(a) * [f(a)]'$, ∵ $f(a') = [f(a)]'$

$= 0^*$

∴ $0^* = f(0)$

(ii) 1 → unit in B

Let 1^* be unit in B'. Then

$f(1) = f(a + a') = f(a) + f(a')$

$= f(a) + [f(a)]'$

$= 1^*$

∴ $1^* = f(1)$ (Proved)

Ex Prove that no Boolean algebra can have three distinct elements

Sol

Soln Let B be a Boolean algebra having three elements. Then B must have two distinct element 0 and 1 as identities w.r.t. '+' & '*'. Let 'a' be the third element of B. Since B is a Boolean algebra, ∴ ∃ an element a' in B s.t.

$$a + a' = 1, \quad a * a' = 0$$

we have three cases–

(i) $a' = a$ (ii) $a' = 0$ (iii) $a' = 1$

(i). If $a' = a$ then $a + a' = 1 \Rightarrow a + a = 1$
$\Rightarrow a = 1$

and $a * a' = 0 \Rightarrow a * a = 0 \Rightarrow a = 0$

But 'a' is different from '0' & '1'

∴ $a' = a$ is not possible.

(ii) If $a' = 0$ then $a + a' = 1 \Rightarrow a + 0 = 1$
$\Rightarrow a = 1$

not possible

(iii) If $a' = 1$ then $a * a' = 0 \Rightarrow a * 1 = 0$
$\Rightarrow a = 0$

not possible.

∴ B can not have three distinct elements